EXTENSIONS OF THE ALTERNATING GROUP OF DEGREE 6 IN THE GEOMETRY OF K3 SURFACES

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ABSTRACT. We shall determine the uniquely existing extension of the alternating group of degree 6 (being normal in the group) by a cyclic group of order 4, which can act on a complex K3 surface.

1. INTRODUCTION

A K3 surface X is a simply-connected compact complex 2-dimensional manifold which admits a nowhere vanishing holomorphic 2-form ω_X. As is well known, K3 surfaces form a 20-dimensional analytic family.

In our previous note [KOZ], we have shown, among all K3 surfaces, the unique existence of the triplet $(F, \tilde{A}_6, \rho_F)$ of a complex K3 surface F and its (faithful) finite group action $\rho_F : \tilde{A}_6 \times F \longrightarrow F$ of $\tilde{A}_6$ on F, up to isomorphisms. The group $\tilde{A}_6$ is an extension of A_6 (being normal in $\tilde{A}_6$) by μ_4, which has also been shown to be the unique maximum finite extension of A_6 in the automorphism groups of $K3$ surfaces. (Here and hereafter, we shall employ the notation of groups as in the list of notation at the end of Introduction.) We have also described the target K3 surface F; it is isomorphic to the minimal resolution of the surface in $\mathbf{P}^1 \times \mathbf{P}^2$ given by the following equation, where $([S : T], [X : Y : Z])$ are the homogeneous coordinates of $\mathbf{P}^1 \times \mathbf{P}^2$:

$$S^2(X^3 + Y^3 + Z^3) - 3(S^2 + T^2)XYZ = 0.$$

However, as was remarked in [KOZ], the action of $\tilde{A}_6$ is so invisible in the equation that it seems hard to find the abstract group structure of the (uniquely existing) group $\tilde{A}_6$ from the equation above and it remains unsolved to date.

The aim of this short note is to describe the group structure of $\tilde{A}_6$ explicitly as an abstract group (Theorem (4.1)).

In contrast to the fact that $\mathrm{Out}(A_n) \simeq C_2$ when $n \geq 3$ and $n \neq 6$, the very special nature of A_6, that *the outer automorphism group $\mathrm{Out}(A_6)$ is isomorphic to a bigger group C_2^2*, makes our humble work non-trivial and interesting. Indeed, corresponding to the three involutions, $\mathrm{Aut}(A_6)$ has three index 2 subgroups $A_6 < G < \mathrm{Aut}(A_6)$, which are S_6, $\mathrm{PGL}(2,9)$ and M_{10}. (See for instance [Su, Chapter 3], [CS, Chapters 10, 11]. See also Section 1). According to Suzuki [Su, Page 300], this extraordinary property of A_6 also makes the classification of simple groups deep and difficult.

We first notice that there are exactly four isomorphism classes of $A_6.\mu_4$ corresponding to the four normal proper subgroups A_6, S_6, $\mathrm{PGL}(2,9)$ and M_{10} of $\mathrm{Aut}\, A_6$ (Theorem (2.3)). This is purely group-theoretic and should be known to

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experts. We then determine which one our $\tilde{A}_6$ is. The part here involves geometric arguments on K3 surfaces such as the representation of the group action on the Picard lattice of a K3 surface and its constraint from projective geometry of a target K3 surface (Proposition (3.2), Lemma (4.2)). It turns out that our $\tilde{A}_6$ is the one which arises from the last normal subgroup M_{10}, the Mathieu group of degree 10 (Theorem (4.1)).

It might be of interest that K3 surfaces could distinguish M_{10} from S_6 and $\mathrm{PGL}(2,9)$ in this way.

Notation.

By S_n, A_n, C_n and $\mu_n \simeq C_n$, we denote the symmetric group of degree n, the alternating group of degree n, the cyclc group of order n, and the multiplicative group of order n (in $\mathbf{C}^\times$) respectively. Then $\mu_n = \langle \zeta_n \rangle$ where $\zeta_n = \exp(2\pi i/n)$. By C_n^m, we denote the direct product of m copies of C_n.

By $\mathrm{PGL}(n,q)$ (resp. $\mathrm{PSL}(n,q)$), we denote the projective linear group (resp. projective special linear group) of $\mathbf{P}(\mathbf{F}_q^{\oplus n})$ over the finite field $\mathbf{F}_q$ of q-elements.

M_{10} is the Mathieu group of degree 10, which is defined to be the pointwise stabilizer subgroup of $\{11, 12\}$ of the Mathieu group M_{12} of degree 12 under the natural action of M_{12} on the twelve element set $\{1, 2, \cdots, 12\}$. (See for instance [CS, Chapters 10, 11]).

We write $G = A.B$ when G fits in the exact sequence

$$1 \longrightarrow A \longrightarrow G \longrightarrow B \longrightarrow 1 \ .$$

(So, for given A and B, there are in general several isomorphism classes of groups of the form $A.B$.) For instance $\mathrm{Aut}(A_6) = A_6.C_2^2$. We always regard A as a normal subgroup of $A.B$.

When the exact sequence above splits, G is a semi-direct product of A and B which we shall write $A : B$. (Again, for given A and B, there are in general several isomorphism classes of groups of the form $A : B$.) We always regard A as a normal subgroup of $A : B$ and B as a subgroup of $A : B$.

By $[G, G]$ and $\mathrm{Cent}(G)$, we denote the commutator subgroup of G and the center of G respectively.

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2. THE ISOMORPHISM CLASSES OF $A_6.\mu_4$

Let G be a group of the form $A_6.\mu_4$, i.e. a group which fits in the exact sequence

$$1 \longrightarrow A_6 \longrightarrow G \xrightarrow{\alpha} \mu_4 \longrightarrow 1 \ .$$

The goal of this section is to determine the isomorphism classes of such G.

In order to make our argument clear, we first remark the following:

Lemma 2.1. *(1) Both G and $\mathrm{Aut}(A_6)$ have exactly one subgroup which is isomorphic to A_6.*
(2) $\mathrm{Aut}(A_6) \simeq A_6.C_2^2$ has exactly one subgroup which is isomorphic to S_6, $\mathrm{PGL}(2,9)$, and M_{10} respectively. These three groups are mutually non-isomorphic even as

abstract groups. In terms of the natural (conjugacy) action on A_6, these three subgroups are also distinguished as follows: S_6 switches the conjugacy classes $5A$ and $5B$ of A_6, but not $3A$ and $3B$; $PGL(2,9)$ switches the conjugacy classes $3A$ and $3B$ of A_6 but not $5A$ and $5B$; and M_{10} switches both. (The notation here is from the Atlas; see also the table in Proposition (3.2).)

So, for instance one can speak of the subgroup M_{10} of $\mathrm{Aut}(A_6)$ without any ambiguity.

Proof. Both G and $\mathrm{Aut}(A_6)$ have a normal subgroup H isomorphic to A_6. This H is the kernel of the natural homomorphism $G \longrightarrow \mu_4$ and $\mathrm{Aut}(A_6) \longrightarrow C_2^2$ respectively. Let K be another subgroup of G or $\mathrm{Aut}(A_6)$ isomorphic to A_6. Since $K \simeq A_6$ is a simple non-commutative group, the homomorphisms $K \longrightarrow \mu_4$ and $K \longrightarrow C_2^2$ are both trivial. Thus $H > K$ and hence $K = H$ by $|K| = |H|$.

Let L be a subgroup of $\mathrm{Aut}(A_6)$ being isomorphic to either S_6, $PGL(2,9)$ or M_{10}. In each case L contains a subgroup isomorphic to A_6. This same A_6 is now the unique subgroup of $\mathrm{Aut}(A_6) = A_6.C_2^2$ and therefore such L corresponds bijectively to the index 2-subgroups of $\mathrm{Aut}(A_6)/A_6 = C_2^2$. Thus, there are exactly three such L. The last two assertions and the fact that $\mathrm{Aut}(A_6)$ contains subgroups isomorphic to S_6, $PGL(2,9)$ and M_{10} are well known. (See for instance [CS, Chap 10] or Atlas table of A_6.)

$\square$

Since A_6 is a normal subgroup of G, it follows that $c(g)(a) = gag^{-1} \in A_6$ for $g \in G$ and $a \in A_6$. We have then a natural group homomorphism

$$c : G \longrightarrow \mathrm{Aut}\,(A_6) \; ; \; g \mapsto c(g) \; ; \; c(g)(a) = gag^{-1} \; .$$

Set

$$N := c(G)$$

and consider the homomorphism:

$$\tilde{c} := (c, \alpha) : G \longrightarrow N \times \mu_4 < \mathrm{Aut}(A_6) \times \mu_4 \; ; \; g \mapsto (c(g), \alpha(g)) \; .$$

We define the natural projections:

$$p_1 : N \times \mu_4 \longrightarrow N \; ; \; (x, y) \mapsto x \; , \; p_2 : N \times \mu_4 \longrightarrow \mu_4 \; ; \; (x, y) \mapsto y \; .$$

Lemma 2.2. *(1) N is either A_6, S_6, $PGL(2,9)$ or M_{10}.*
(2) $\tilde{c}$ is injective.
(3) $\alpha^{-1}(\mu_2) \simeq A_6 \times \mu_2$ and $\tilde{c}(\alpha^{-1}(\mu_2)) = A_6 \times \mu_2$ in $N \times \mu_4$. In particular, if $\tilde{h} \in G$ and $c(\tilde{h}) \in N \setminus A_6$, then $\alpha(\tilde{h}) = \pm\zeta_4$.

Proof. Since A_6 is simple and non-commutative, the restriction $c|A_6$ is injective. Thus

$$A_6 = c(A_6) < N < \mathrm{Aut}(A_6) = A_6.C_2^2 \; .$$

Moreover, we have that $N \neq A_6.C_2^2$; otherwise $c : A_6.\mu_4 \simeq A_6.C_2^2$ is an isomorphism and $\mu_4 \simeq C_2^2$ by Lemma (2.1), a contradiction. Thus, we obtain (1).

Let $g \in \mathrm{Ker}(\tilde{c})$. Then, $\alpha(g) = 1$, i.e. $g \in A_6$. Since A_6 is simple and non-commutative, it follows that $\mathrm{Cent}(A_6) = \{1\}$. Thus $g = 1$ by $c(g) = 1$, and we obtain (2).

Since μ_2 is a normal subgroup of μ_4 of index 2 and α is surjective, it follows that $\alpha^{-1}(\mu_2)$ is a normal subgroup of G of index 2. Thus $[\alpha^{-1}(\mu_2) : A_6] = 2$. Take $g \in G$ such that $\alpha(g) = \zeta_4$. Then $\alpha^{-1}(\mu_2) = \langle A_6, g^2 \rangle$. Since $[c(G) : A_6] \leq 2$ by (1),

we have $c(g^2) \in A_6$. Thus, $c(g^2 h^{-1}) = 1$ for some $h \in A_6$. Set $f := g^2 h^{-1}$. Then $\tilde{c}(f) = (1, -1)$, whence ord $f = 2$ by (2), and $fa = af$ for each $a \in A_6$. Thus

$$\alpha^{-1}(\mu_2) = \langle A_6, f \rangle = A_6 \times \langle f \rangle \simeq A_6 \times \mu_2 \ .$$

This implies

$$A_6 \times \mu_2 < \tilde{c}(\alpha^{-1}(\mu_2)) < N \times \mu_2$$

in $N \times \mu_4$, and hence $A_6 \times \mu_2 = \tilde{c}(\alpha^{-1}(\mu_2))$, because the orders are the same. □

The four candidates for N in Lemma (2.2)(1) give four different group structures on $G = A_6.\mu_4$, which are all semi-direct products, indeed:

Theorem 2.3. *There are exactly four possible group structures of G, up to isomorphism. More explicitly, G is isomorphic to the following subgroup of $N \times \mu_4$ corresponding to each of the four candidates for $N < Aut(A_6)$ as in (2.2)(1):*

(1) If $N = A_6$, then $G = A_6 \times \mu_4$. In this case, we set $\tilde{g} = (1, \zeta_4)$. Then $\tilde{g}$ is an order 4 element and $G = A_6 \times \langle \tilde{g} \rangle$.

(2) If $N = S_6$, then $G = A_6.\langle \tilde{g} \rangle = A_6 : \langle \tilde{g} \rangle$, where $\tilde{g} = (g, \zeta_4)$ and $g = (1, 2) \in S_6$.

(3) If $N = PGL(2, 9)$, then $G = A_6.\langle \tilde{g} \rangle = A_6 : \langle \tilde{g} \rangle$, where $\tilde{g} = (h^5, \zeta_4)$ for some element $h \in PGL(2, 9)$ of order 10.

(4) If $N = M_{10}$, then $G = A_6.\langle \tilde{g} \rangle = A_6 : \langle \tilde{g} \rangle$, where $\tilde{g} = (g, \zeta_4)$ and g is an order 4-element of $M_{10} \setminus A_6$.

In each case, the semi-direct product structure is the natural one. We denote the groups in (1), (2), (3), (4) by $A_6(4)$, $S_6(2)$, $PGL(2, 9)(2)$, $M_{10}(2)$ respectively.

Proof. It is clear that the four groups $A_6(4)$, $S_6(2)$, $PGL(2, 9)(2)$, $M_{10}(2)$ satisfy the required conditions, i.e. G is of the form $A_6.\mu_4$ and $c(G) = N$.

Let us next show that G is isomorphic to the group $A_6(4)$, $S_6(2)$, $PGL(2, 9)(2)$, $M_{10}(2)$, if $N = A_6$, S_6, $PGL(2, 9)$, M_{10} respectively.

If $c(G) = A_6$, then $\tilde{c} : G \longrightarrow A_6 \times \mu_4$ is an isomorphism. Indeed, $\tilde{c}$ is injective by Lemma (2.2)(2) and $|G| = |A_6 \times \mu_4|$.

Consider the case $N = S_6$ or $PGL(2, 9)$. Since $c(G) = N$, there is an element $\tilde{g}$ of G such that $c(\tilde{g}) \in N \setminus A_6$, ord$(c(\tilde{g})) = 2$ and $\alpha(\tilde{g}) = \pm\zeta_4$. Indeed, by Lemma (2.2)(3), one can take a preimage of g in (2) and (3) as $\tilde{g}$. Moreover, $\zeta_4 \mapsto -\zeta_4$ gives a group automorphism of μ_4, the isomorphism class of G does not depends on the choice of the sign of $\alpha(\tilde{g})$. Therefore we may also adjust as $\alpha(\tilde{g}) = \zeta_4$ for a chosen g. Since $\tilde{c}$ is injective, this also implies ord$(\tilde{g}) = 4$ and consequently $G = A_6 : \langle \tilde{g} \rangle$ as claimed.

Finally consider the case $N = M_{10}$. Note that $M_{10} \setminus A_6$ has no order 2 element and order 4 elements of $M_{10} \setminus A_6$ form one conjugacy class of M_{10}. (See for instance, [Atlas table of A_6].) Let $g \in M_{10} \setminus A_6$ be an order 4 element and $\tilde{g} \in G$ be an element such that $c(\tilde{g}) = g$. Then $\alpha(\tilde{g}) = \pm\zeta_4$ by Lemma (2.2)(3). Now, as in the cases (2) and (3), we may adjust as $\alpha(\tilde{g}) = \zeta_4$ for a chosen g, and $G = A_6 : \langle \tilde{g} \rangle$ as claimed in (4).

Since A_6 is the unique subgroup of G, the image N of the homomorphism c is uniquely determined by G. Thus, the four groups $A_6(4)$, $S_6(2)$, $PGL(2, 9)(2)$, $M_{10}(2)$ are not isomorphic to one another.

Now we are done.

□

3. Some lattice representations

In this section, we recall some known facts about K3 surface from [BPV] and about K3 surfaces admitting an A_6-action from [KOZ]. For details, please refer to these references and references therein.

By a $K3$ surface, we mean a simply-connected compact complex surface X admitting a nowhere vanishing global holomorphic 2-form ω_X. K3 surfaces form a 20-dimensional analytic family. The second cohomology group $H^2(X, \mathbf{Z})$ together with its cup product becomes an even unimodular lattice of index $(3, 19)$ and is isomorphic to the so-called $K3$ lattice

$$L := U^{\oplus 3} \oplus E_8^{\oplus 2} \ ,$$

where E_8 is the negative definite even unimodular lattice of rank 8. We denote by $S(X)$ the Néron-Severi lattice of X. This is a primitive sublattice of $H^2(X, \mathbf{Z})$ generated by the (first Chern) classes of line bundles. The rank of $S(X)$ is called the Picard number of X and is denoted by $r(X)$. We have $0 \leq r(X) \leq 20$. Denote by $T(X)$ the transcendental lattice of X, i.e. the minimum primitive sublattice whose $\mathbf{C}$-linear extension contains the class ω_X, or equivalently $T(X) = S(X)^{\perp}$ in $H^2(X, \mathbf{Z})$. If X is projective, then $S(X)$ is of index $(1, r(X) - 1)$ (and vice versa), $S(X) \cap T(X) = \{0\}$ and $S(X) \oplus T(X)$ is a finite-index sublattice of $H^2(X, \mathbf{Z})$.

Let (X, G, ρ) be a triplet consisting of a $K3$ surface X, a finite group G and a faithful action $\rho : G \times X \longrightarrow X$. Then G has a 1-dimensional representation on $H^0(X, \Omega_X^2) = \mathbf{C}\omega_X$ given by $g^*\omega_X = \alpha(g)\omega_X$, and we have an exact sequence, called the basic sequence:

$$1 \longrightarrow G_N := \operatorname{Ker} \alpha \longrightarrow G \overset{\alpha}{\longrightarrow} \mu_I \longrightarrow 1 \ .$$

The importance of the basic sequence was first noticed by Nikulin [Ni]. We call G_N the symplectic part and $\mu_I := \langle \zeta_I \rangle$ (resp. I), the transcendental part (resp. the transcendental value) of the action ρ. We note that if $A_6.\mu_4$ acts faithfully on a K3 surface then $G_N \simeq A_6$ and the transcendental part is isomorphic to μ_4. This follows from the fact that A_6 is simple and also maximum among all finite groups acting on a K3 surface symplectically. This is a result of Mukai [Mu]. We also note that X is projective if $I \geq 2$ [Ni].

We say that 2 triplets (X, G, ρ) and (X', G', ρ') are isomorphic if there are a group isomorphism $f : G' \simeq G$ and an isomorphism $\varphi : X' \simeq X$ such that the following diagram commutes:

$$
\begin{array}{ccc}
G \times X & \overset{\rho}{\longrightarrow} & X \\
{\scriptstyle f \times \varphi}\uparrow & & \uparrow {\scriptstyle \varphi} \\
G' \times X' & \overset{\rho'}{\longrightarrow} & X'
\end{array}
$$

The main result of [KOZ] is the following:

Theorem 3.1. *Let G be a finite group acting faithfully on a K3 surface X. Assume that $A_6 < G$ and $I \geq 2$, where I is the transcendental value. Then we have $G_N = A_6$, $I = 2$ or 4, and $\operatorname{rank} L^{G_N} = 3$. In particular, $S(X)^{G_N} = \mathbf{Z}H$, where H is an ample class, and $\operatorname{rank} T(X) = 2$.*

Consider the case having maximum $I = 4$, i.e. the case where G, which we shall denote by $\tilde{A}_6$, is a group of the form $A_6.\mu_4$. Then there is a unique triplet $(F, \tilde{A}_6, \rho_F)$ consisting of a K3 surface F and a faithful group action $\rho_F : \tilde{A}_6 \times F \longrightarrow F$ of $\tilde{A}_6$ on F up to isomorphism. In particular, the isomorphism class of

$\tilde{A}_6$ is unique. Moreover the K3 surface F has Picard number 20 and is uniquely characterized by the following equivalent conditions:

(1) F is the K3 surface whose transcendental lattice $T(F) = \mathbf{Z}\langle t_1, t_2 \rangle$ has the intersection matrix
$$\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} .$$

(2) F is the minimal resolution of the double cover $\overline{F}$ of the (rational) elliptic modular surface E with level 3 structure. The double cover is branched along two of a total of 4 singular fibres of the same type I_3 and $\overline{F}$ has 6 ordinary double points.

(3) F is the minimal resolution of the surface in $\mathbf{P}^1 \times \mathbf{P}^2$ given by the following equation, where $([S:T], [X:Y:Z])$ are coordinates of $\mathbf{P}^1 \times \mathbf{P}^2$:
$$S^2(X^3 + Y^3 + Z^3) - 3(S^2 + T^2)XYZ = 0.$$

In the course of proof, we have also shown the following fact. This will also be crucial in the next section and will be proved again here:

Proposition 3.2. *Let X be a K3 surface of Picard number 20. Assume that X admits a faithful action of $G_N = A_6$. Then, as A_6-modules, one has the irreducible decomposition*
$$S(X) \otimes \mathbf{C} = \chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \chi_6 .$$

In this description, we used Atlas notation for irreducible characters/representations of A_6 as in the Table below.

	1A	2A	3A	3B	4A	5A	5B
χ_1	1	1	1	1	1	1	1
χ_2	5	1	2	-1	-1	0	0
χ_3	5	1	-1	2	-1	0	0
χ_4	8	0	-1	-1	0	$(1-\sqrt{5})/2$	$(1+\sqrt{5})/2$
χ_5	8	0	-1	-1	0	$(1+\sqrt{5})/2$	$(1-\sqrt{5})/2$
χ_6	9	1	0	0	1	-1	-1
χ_7	10	-2	1	1	0	0	0

Proof. Since many ingredients and some idea in the next section have already appeared in the proof of this proposition, we prove this proposition for reader's convenience.

Recall that the order structure of A_6 is as follows:

order[conjugacy class]	1 [1A]	2 [2A]	3 [3A]	3 [3B]	4 [4A]	5 [5A]	5 [5B]
cardinality	1	45	40	40	90	72	72

Moreover, by [Ni], the number of the fixed points of the symplectic action is as follows.

$ord(g)$	1	2	3	4	5	6	7	8
$|X^g|$	X	8	6	4	4	2	3	2

Set $\tilde{H}(X,\mathbf{Z}) = H^0(X,\mathbf{Z}) \oplus H^2(X,\mathbf{Z}) \oplus H^4(X,\mathbf{Z})$. Now, by applying the topological Lefschetz fixed point formula for $G_N = A_6$, we calculate that

$$\operatorname{rank} \tilde{H}(X,\mathbf{Z})^{A_6} = \frac{1}{|A_6|} \sum_{g \in A_6} \operatorname{tr}(g^*|\tilde{H}(X,\mathbf{Z}))$$

$$= \frac{1}{360}(24 + 8 \cdot 45 + 6 \cdot 80 + 4 \cdot 90 + 4 \cdot 144) = 5 .$$

Since G_N is trivial on $H^0(X,\mathbf{Z})$, $H^4(X,\mathbf{Z})$ and $T(X)$, one has $S(X)^{A_6} = \mathbf{Z}H$ and H is an ample primitive class of X. (Here we used the fact that any K3 surface of Picard number 20 is projective, because $S(X) \otimes \mathbf{R} = H^{1,1}(X,\mathbf{R})$ and therefore $S(X)$ is of signature $(1,19)$ and the fact that if h is an ample class of X and G is a finite group acting on X, then $\sum_{g \in G} g^* h$ is also an ample class which is invariant under G.)

Thus the irreducible decomposition of $S(X)$ by A_6 must be of the following form:

$$S(X) \otimes \mathbf{C} = \chi_1 \oplus a_2 \chi_2 \oplus a_3 \chi_3 \oplus a_4 \chi_4 \oplus a_5 \chi_5 \oplus a_6 \chi_6 \oplus a_7 \chi_7 ,$$

where a_i are non-negative integers. Let us determine a_i's. As in (2), using the topological Lefschetz fixed point formula and the fact that $\operatorname{rank} T(X) = 2$, we have

$$\chi_{\mathrm{top}}(X^g) = 4 + \operatorname{tr}(g^*|S(X))$$

for $g \in A_6$. Running g through the 7-conjugacy classes of A_6 and calculating both sides based on Nikulin's table and the character table above, we obtain the following system of equations:

$$20 = 1 + 5(a_2 + a_3) + 8(a_4 + a_5) + 9a_6 + 10a_7 ,$$
$$4 = 1 + (a_2 + a_3) + a_6 - 2a_7 ,$$
$$2 = 1 + (2a_2 - a_3) - (a_4 + a_5) + a_7 , \quad 2 = 1 + (-a_2 + 2a_3) - (a_4 + a_5) + a_7 ,$$
$$0 = 1 - (a_2 + a_3) + a_6 ,$$
$$0 = 1 + (\frac{1 - \sqrt{5}}{2}a_4 + \frac{1 + \sqrt{5}}{2}a_5) - a_6 , \quad 0 = 1 + (\frac{1 + \sqrt{5}}{2}a_4 + \frac{1 - \sqrt{5}}{2}a_5) - a_6 .$$

Now, we get the result by solving this system of Diophantine equations. $\qquad\square$

4. Determination of the isomorphism class of $\tilde{A}_6$

Let $\tilde{A}_6$ be a group of the form $A_6.\mu_4$ which can act on a K3 surface X. Among four candidates $A_6(4)$, $S_6(2)$, $\mathrm{PGL}(2,9)(2)$, and $M_{10}(2)$ for $\tilde{A}_6$ (Theorem (2.3)), only one is isomorphic to $\tilde{A}_6$ (Theorem (3.1)).

The aim of this section is to determine the isomorphism class of this $\tilde{A}_6$:

Theorem 4.1. $\tilde{A}_6$ *is isomorphic to the group* $M_{10}(2)$.

Proof. Set $G = \tilde{A}_6$. It suffices to show that G is not isomorphic to $A_6(4)$, $S_6(2)$, $\mathrm{PGL}(2,9)(2)$. Suppose to the contrary that G is isomorphic to one of these three groups. Let $\tilde{g} \in G$ be an order 4 element chosen in Theorem (2.3). Set $\iota := \tilde{g}^2$. Then, by the description of $\tilde{g}$, we have $G = A_6 : \langle \tilde{g} \rangle$ and $c(\iota) = 1$, i.e. $\iota a = a\iota$ for all $a \in A_6$, and $\alpha(\iota) = -1$. Thus $\iota^* \omega_X = -\omega_X$ on a target K3 surface X.

First we prove the result below to get some geometric constraint of the pair $(X, \tilde{A}_6)$.

Lemma 4.2. *(1)* $\chi_{top}(X^\iota) \leq 0$. *Moreover,* $\chi_{top}(X^\iota) = 0$ *if and only if* $X^\iota = \emptyset$.

(2) *Let σ be an element of A_6. Assume that the order of σ is either 3 or 5. Then*
$$\chi_{top}(X^{\iota\sigma}) \geq 0.$$
    Here $\chi_{top}(X^a)$ is the topological Euler number of the fixed locus X^a of $a \in G$.

Proof. Since $\iota^*\omega_X = -\omega_X$, the action of ι on X is locally linearlizable at $P \in X^\iota$ as

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Set $C := X^\iota$. If it is empty, then $\chi_{top}(C) = 0$. Assume that $C \neq \emptyset$. Then C is a smooth curve, possibly reducible. Since $\iota a = a\iota$ for each $a \in A_6$ (by $c(\iota) = 1$), the curve C is stable under the action of A_6. Thus the class $[C] \in S(F)$ is A_6-invariant. Since $S(X)^{A_6} = \mathbf{Z}H$ and H is an ample primitive class, C is also an ample class. Therefore, C is connected and hence irreducible by the smoothness. Since $(C^2) > 0$, we have by the adjunction formula

$$\chi_{top}(C) = 2 - 2g(C) = -(K_X + C.C) = -(C^2) < 0 .$$

Let us show (2). Assume that σ is of order p, where p is either 3 or 5. Set $\tau = \iota\sigma$. Then $X^\tau \subset X^{\tau^2}$. We have $\tau^2 = \sigma^2 \in A_6 = G_N$ (and is of order p) by $\iota\sigma = \sigma\iota$. Thus, the set X^{τ^2} is a finite set as in Nikulin's table in Proposition (3.2). Hence X^τ is also a finite set, and therefore $\chi_{top}(X^\tau) \geq 0$.

$\square$

Let us recall the irreducible decomposition of $S(X)$ in Proposition (3.2). Since X has an ample G-invariant class, we have $\tilde{g}^*(H) = H$ and hence $\tilde{g}^*|\chi_1 = id$. Since $G = A_6 : \langle \tilde{g} \rangle$, we have also $\tilde{g}^*(\chi_6) = \chi_6$ and either $\tilde{g}^*(\chi_2) = \chi_2$ and $\tilde{g}^*(\chi_3) = \chi_3$ or $\tilde{g}^*(\chi_2) = \chi_3$ and $\tilde{g}^*(\chi_3) = \chi_2$. Since $\iota = \tilde{g}^2$, we have $\iota^*(\chi_i) = \chi_i$ for each $i = 1, 2, 3, 6$. Since $\iota a = a\iota$ for all $a \in A_6$, it follows that $\iota^*|\chi_i$ are scalar multiplications by Schur's lemma. Moreover, since ι is of order 2, we have

$$\iota^*|\chi_1 = 1 \ , \ \iota^*|\chi_i = (-1)^{n_i} id_{\chi_i}$$

for some $n_i \in \mathbf{Z}$ for each $i = 2, 3, 6$. We have also that $\iota^*|H^0(X) \oplus H^4(X) = id$, $\iota^*|T(X) = -id_{T(X)}$ and $\text{rank}\,T(X) = 2$. Thus, by the topological Lefschetz formula, we obtain

$$\chi_{top}(X^\iota) = 1 + 5 \cdot ((-1)^{n_2} + (-1)^{n_3}) + 9 \cdot (-1)^{n_6} \ --- \ (*) .$$

The value $\chi_{top}(X^\iota)$ must be non-positive by Lemma (4.2)(1), whence

$$((-1)^{n_2}, (-1)^{n_3}, (-1)^{n_6})$$

must be either one of:

$$(-1, 1, -1) \ , \ (1, -1, -1) \ , \ (-1, -1, -1) \ , \ (-1, -1, 1) .$$

Consider first the case $(-1, 1, -1)$ (resp. $(1, -1, -1)$). Take an order 3 element σ of A_6 from the conjugacy class 3A (resp. 3B). Note that $(\iota\sigma)^*|H^0(X) \oplus H^4(X) = id$, $(\iota\sigma)^*|T(X) = -1$ and $\text{rank}\,T(X) = 2$. Then by the topological Lefschetz formula and by the character table, we calculate

$$\chi_{top}(X^{\iota\sigma}) = \text{tr}((\iota\sigma)^*|S(X)) = 1 - 2 - 1 + 0 = -2 < 0 ,$$

a contradiction to Lemma (4.2)(2).

Consider next the case $(-1, -1, -1)$. Since $\tilde{g}^*(\chi_6) = \chi_6$ and $\iota^*|\chi_6 = -id_{\chi_6}$, it follows that the eigenvalues of $\tilde{g}^*|\chi_6 = \pm\zeta_4$ and $\text{tr}(\tilde{g}^*|\chi_6) = (9 - 2n)\zeta_4$, where n is the multiplicity of $-\zeta_4$. Note that $\text{tr}(\tilde{g}^*|T(X)) = \zeta_4 - \zeta_4 = 0$.

So, if $\tilde{g}^*(\chi_2) = \chi_3$, then $\mathrm{tr}(\tilde{g}^*|\chi_2 \oplus \chi_3) = 0$. Thus

$$\chi_{\mathrm{top}}(X^{\tilde{g}}) = 2 + \mathrm{tr}(\tilde{g}^*|S(X)) = 3 + (9 - 2n) \cdot \zeta_4 \notin \mathbf{Z} ,$$

a contradiction to the obvious fact that $\chi_{\mathrm{top}}(X^{\tilde{g}}) \in \mathbf{Z}$.

If $\tilde{g}^*(\chi_2) = \chi_2$, then for the same reason as above, the eigenvalues of

$$\tilde{g}^*|\chi_2 \oplus \chi_3 \oplus \chi_6$$

are $\pm\zeta_4$. Let n be the multiplicity of $-\zeta_4$. Then, since

$$\dim \chi_2 \oplus \chi_3 \oplus \chi_6 = 19 ,$$

we have

$$\chi_{\mathrm{top}}(X^{\tilde{g}}) = 2 + \mathrm{tr}(\tilde{g}^*|S(X)) = 3 + (19 - 2n)\zeta_4 \notin \mathbf{Z} ,$$

again a contradiction.

Finally consider the case $(-1, -1, 1)$.

Let us first treat the cases where G is isomorphic to $A_6(4)$ or $S_6(2)$. By the formula (*), we have $\chi_{\mathrm{top}}(X^{\iota}) = 0$ and therefore $X^{\iota} = \emptyset$ by Lemma (4.2)(1). Let $\tau := (456)$ in A_6. Then, by the shape of $\tilde{g}$, we have $\tau\tilde{g} = \tilde{g}\tau$ in G. Then $\tilde{g}$ acts on X^{τ}. Since X^{τ} is a 6 element set (see Nikulin's table in Section 3) and $\tilde{g}$ is of order 4, it follows that $\iota(= \tilde{g}^2)$ fixes at least two points in X^{τ}. In particular, $X^{\iota} \neq \emptyset$, a contradiction.

Let us next consider the case where G is isomorphic to $\mathrm{PGL}(2, 9)(2)$.

Set $V := (S(X) \otimes \mathbf{Q})^{\iota^*}$. Then V is A_6-stable by $\iota a = a\iota$ for all $a \in A_6$. By $(n_1, n_2, n_3) = (-1, -1, 1)$ and by the fact that the action ι^* is defined over $S(X) \otimes \mathbf{Q}$, we then have

$$V \otimes \mathbf{C} = \chi_1 \oplus \chi_6 .$$

Recall that $\chi_1 = \mathbf{C}H$, where H is an ample A_6-invariant class. Consider the orthogonal complement $L := H_V^{\perp}$ of H in V. Then L is also A_6-stable and satisfies

$$L \subset S(X) \otimes \mathbf{Q} \text{ and } L \otimes \mathbf{C} = \chi_6 .$$

Since $c(\tilde{g}) = g$ switches the conjugacy classes $3A$ and $3B$ of A_6, while $\chi_2(3A) = 2 \neq -1 = \chi_2(3B)$, we have

$$\tilde{g}^*(\chi_2) = \chi_3 , \ \tilde{g}^*(\chi_3) = \chi_2 ,$$

and therefore

$$\mathrm{tr}\,(\tilde{g}^*|\chi_2 \oplus \chi_3) = 0 .$$

Moreover, since $\iota = \tilde{g}^2$ and $\iota^*|\chi_6 = id$, we have a matrix representation

$$\tilde{g}^*|\chi_6 = I_{9-s} \oplus (-I_s)$$

under certain *rational* basis $\langle u_i \rangle_{i=1}^9$ of L. This is because $\tilde{g}^*$ is defined over $S(X)$ (so that $\tilde{g}^*|\chi_6$ is defined over L) and the eigenspace decomposition of $\tilde{g}^*|\chi_6$ of eigenvalues $\pm 1 \in \mathbf{Q}$ are also rationally defined. We have also that

$$\mathrm{tr}\,(\tilde{g}^*|\chi_6) = 9 - 2s .$$

Note also that

$$\mathrm{tr}(g^*|T(X)) = \zeta_4 - \zeta_4 = 0 ,$$
$$\mathrm{tr}(g^*|H^0(X) \oplus H^4(X)) = 1 + 1 = 2 .$$

Thus, by the topological Lefschetz fixed point formula, one calculates

$$\chi_{\mathrm{top}}(X^{\tilde{g}}) = 2 + 1 + (9 - 2s) = 12 - 2s .$$

Since $X^\iota = \emptyset$, we have $X^{\tilde{g}} = \emptyset$ as well by $\iota = \tilde{g}^2$. Therefore, $\chi_{\text{top}}(X^{\tilde{g}}) = 0$ and $s = 6$.

Let us recall that $\tilde{c}(\tilde{g}) = (h^5, \zeta_4)$ for some element $h \in \text{PGL}(2,9)$ of order 10. Then, $h^2 \in A_6$ (by $[\text{PGL}(2,9) : A_6] = 2$) and therefore there is an element $\sigma \in A_6(< G)$ such that $\tilde{c}(\sigma) = (h^2, 1)$. Using the injectivity of $\tilde{c}$, we see that σ is an element of order 5 and satisfies $\sigma\tilde{g} = \tilde{g}\sigma$. Note that σ^* is defined over $S(X)$ and therefore $\sigma^*|\chi_6$ is defined over L. Thus, under the same *rational* basis $\langle u_i \rangle_{i=1}^9$ of L, we have a *rational* matrix representation

$$\sigma^*|\chi_6 = A \oplus B ,$$

where $A \in \text{GL}(3, \mathbf{Q})$ and $B \in \text{GL}(6, \mathbf{Q})$.

Since $\sigma \in A_6$ is of order 5, we have $\text{tr}(\sigma^*|\chi_6) = -1$ by the definition of χ_6. Thus

$$\text{tr}A + \text{tr}B = -1 .$$

On the other hand, since σ is of order 5, the eigenvalues of A and B are all root of 5. Since A and B are rational matrices of orders 3 and 6 and since $\varphi(5) = 4$, we have then

$$\text{tr}\, A = 1 + 1 + 1 = 3 ,$$

and either

$$\text{tr}\, B = 1 + 1 + \sum_{i=1}^{4} \zeta_5^i = 1 ,$$

or

$$\text{tr}\, B = 1 + 1 + 1 + 1 + 1 + 1 = 6 .$$

However, then

$$\text{tr}A + \text{tr}B = 4 \text{ or } 9 ,$$

a contradiction to the previous equality.

This completes the proof of Theorem (4.1).

$\square$

Remark 4.3. In the above proof, the last case $(-1, -1, 1)$ can be ruled out geometrically. Indeed, in this case we have $X^\iota = \emptyset$ and hence $X^{\tilde{g}} = \emptyset$. This means that the surface X admits a free action of a cyclic group of order 4, which is impossible because no K3 surface may admit such an action (the algebraic Euler number of a K3 surface is equal to 2, not divisible by 4).

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